Cuntz semigroups of C^*-algebras of stable rank one and projective Hilbert modules

Huaxin Lin

Abstract

Let A be a simple C^*-algebra of stable rank one and let p and q be two σ-compact open projections. It is proved that there is a continuous path of unitaries in $\tilde{A}$ which connects open sub-projections of p which is compactly contained in p to those in q. It is also shown that every Hilbert module is projective in the category whose morphisms are bounded module maps with adjoints. A discussion of projective Hilbert modules (whose morphisms are bounded module maps) is also given.

Recently the Cuntz semigroups of C^*-algebras have attracted some previously unexpected attention. The Cuntz relations for positive elements in a C^*-algebra was introduced by J. Cuntz (see [9]). The Cuntz semigroups, briefly, are semigroups of equivalence classes of positive elements in a C^*-algebras. This relation is similar to the Murry and Von Neumann equivalence relation for projections. The renew interests in the Cuntz semigroups probably begins with Toms's example ([20]) which shows that two unital simple AH-algebras with the same traditional Elliott invariant may have different Cuntz semigroups. It is a hope of many that the Cuntz semigroups may be used in the classification of amenable C^*-algebras. This note limits itself to the clarification of a couple issues related to the Cuntz semigroups and its relation with Hilbert modules.

While the Cuntz semigroups may be useful tools to distinguish some C^*-algebras, they are not necessarily easy to compute in general. One problem is that the Cuntz semigroup is not a homotopy invariant. Let A be a C^*-algebra and let $f \in C([0,1], A)$ so that $f(t) \geq 0$ for all $t \in [0,1]$. One easily sees that $f(0)$ and $f(1)$ are unlikely related in the Cuntz relation. On the other hand, Cuntz introduced several versions of the relation among positive elements in C^*-algebras. These relations also give equivalence relations among open projections of C^*-algebras. It will be presented, following a result of L. G. Brown, if two σ-compact open projections are homotopy, then they are actually equivalent in (a strong) Cuntz relation.

Another homotopy question is whether two positive elements are homotopy in a suitable sense if they are equivalent in the sense of Cuntz. Under the assumption that A is simple and has stable rank one, it is shown in this note that two σ-compact open projections are Cuntz equivalent if and only if there is a continuous path of unitaries $\{u(t) : t \in [0,1)\}$ which connects these two open projections in the sense that will be described in 2.6 and 2.9. In particular, any pre-compact open subprojection (see 2.7) of p is unitarily equivalent to a pre-compact open subprojection of q.

Let A be C^*-algebra and let $a, b \in A \otimes \mathcal{K}$. Then $H_1 = \overline{aA}$ and $H_2 = \overline{bA}$ are two Hilbert A-modules. Suppose that p_a and p_b are range projections of a and b in $(A \otimes \mathcal{K})^{**}$. Then p_a and p_b are Cuntz equivalent (see 1.1) if and only if H_1 and H_2 are isomorphic as Hilbert A-modules. So Hilbert modules and the Cuntz semigroups are closely related. In this note, using another result of L. G. Brown, it is shown that p_a is dominated by p_b in the sense of Cuntz if and only if there is a bounded module map $T : H_2 \to H_1$ (which may not have an adjoint) whose range is dense in H_1. Projectivity of Hilbert modules have been recently brought into attention. In

the last section of this note, quite differently from the pure algebraic analogy, it is shown that every Hilbert module over a C^*-algebra A is projective in the category of Hilbert A-modules with bounded module maps with adjoints as morphisms. However, for Hilbert modules over C^*-algebra A, sometime the category of Hilbert A-modules with bounded module maps (may or may not have adjoints) is also useful. To determine which Hilbert A-modules are projective in that category is more difficult. A discussion on this problem will also be presented.

1 The Cuntz Semigroups

Definition 1.1. Let A be a C^*-algebra and let $a \in A_+$. Denote by $Her(a) = \overline{aAa}$ the hereditary C^*-subalgebra of A generated by a. Denote by p_a the range projection of a in A^{**}. It is an open projection of A in A^{**}. Dnote $Her(p) = pA^{**}p \cap A = Her(a)$.

Suppose that $a, b \in A_+$. One writes $a \lessapprox b$ if there exists $x \in A$ such that $x^*x = a$ and $xx^* \in Her(b)$. One writes $a \lesssim b$, if there exists a sequence $r_n \in A$ such that $r_n^* b r_n \to a$ in norm. If $a \lesssim b$ and $b \lesssim a$, then one writes $a \sim b$. The relation "$\sim$" is an equivalence relation. The equivalence class represented by a will be written as $\langle a \rangle$. Denote by $W(A)$ the equivalence classes of positive elements in $M_\infty(A)$ with respect to "$\sim$". So

$$W(A) = \{\langle a \rangle : a \in M_\infty(A)\}.$$

The semigroup $W(A)$ is called the Cuntz semigroup. One can also define the same relation in $A \otimes \mathcal{K}$. The corresponding semigroup is denoted by $Cu(A)$.

Let p and q be two open projections of $M_n(A)$ in $M_n(A)^{**} = (M_n(A^{**}))$ for some integer $n \geq 1$ (or let p and q be two open projections of $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on l^2). One says that p and q are Cuntz equivalent and writes $p \approx_{cu} q$, if there exists a partial isometry $v \in M_n(A)^{**}$ (or $v \in (A \otimes \mathcal{K})^{**}$) such that

$$v^*v = p, \ vv^* = q \ \text{ and } \ vav^* \in Her(q) \ \text{ for all } \ a \in Her(p).$$

The relation "$\approx_{cu}$" is also an equivalence relation. The equivalence class represented by p will be denoted by $[p]$. An open projection of A is said to be σ-compact, if $p = p_a$ for some $a \in A_+$. Denote by $Co(A)$ the equivalence classes of σ-compact open projections of $M_n(A)$ in $M_n(A)^{**}$ for all $n \geq 1$. Denote by $Co(A \otimes \mathcal{K})$ the equivalence classes of σ-compact open projections of $A \otimes \mathcal{K}$ in $(A \otimes \mathcal{K})^{**}$. Note that $M_n(A \otimes \mathcal{K}) \cong A \otimes \mathcal{K}$. These also form semigroups. One write $[p] \leq [q]$, if $p \approx_{cu} q'$ for some open projection $q' \leq q$.

Let $a, b \in M_\infty(A)$ be two positive elements. Then $[p_a] \leq [p_b]$ if and only if $a \lessapprox b$. One writes $p \sim_{cu} q$ if $[p] \leq [q]$ and $[q] \leq [p]$. This is also an equivalence relation. Denote by $\langle p \rangle$ the equivalence class represented by p.

These relations were first introduced by Cuntz (see [9]) and the readers are referred to [9], [10], [2], [18] and [12] for more details.

There are significant differences between $W(A)$ and $Co(A)$ (and differences between ($Cu(A)$ and $Co(A \otimes \mathcal{K})$) in general. An example that $W(A) \neq Co(A)$ for stably finite C^*-algebra was given in [6]. Let A be a purely infinite simple C^*-algebra and let a, $b \in A_+ \setminus \{0\}$. Then $\langle a \rangle = \langle b \rangle$. Thus $Cu(A)$ contains only zero and one other element. It is not quite useful in this case. On the other hand, it follows from a result of S. Zhang [21] (see also Cor. 11 of [13]) that $Her(a)$ and $Her(b)$ are stable and isomorphic if neither p_a nor p_b are in A. In fact the isomorphism can be given by an isometry. From this, one easily obtains the following.

Proposition 1.2. *Let A be a purely infinite simple C^*-algebra Then*

$$Co(A) = V(A) \sqcup \{\infty\},$$

where $V(A)$ is the Murry-Von Neumann equivalence classes of projections in $M_\infty(A)$ and ∞ is represented by a non-zero σ-compact open projection which is not in A.

However, $W(A)$ and $Co(A)$ could be often the same.

Definition 1.3. Let $\epsilon > 0$. Define

$$f_\epsilon(t) = \begin{cases} 0 & \text{if } t \in [0, \epsilon/2] \\ \text{linear} & \text{if } t \in [\epsilon/2, \epsilon] \\ 1 & \text{if } t \in [\epsilon, \infty). \end{cases}$$

Lemma 1.4. (G. K. Pedersen (Theorem 5 of [17])) *Let A be a C^*-algebra with stable rank one. Suppose that $x \in A$. Then, for each $t \in (0, \|x\|]$, there is a unitary $u_t \in \tilde{A}$ such that*

$$u_t^* p_t u_t = q_t,$$

where p_t is the open spectral projection of $|x|$ associated with $(t, \|x\|]$ and q_t is the open spectral projection of $|x^*|$ associated with $(t, \|x\|]$, respectively. Moreover,

$$u_{t'} p_t = u_t p_t \text{ and } u_{t'} p_t u_{t'} = q_t$$

for all $0 < t' < t < \|x\|$.

Proof. Note, by Theorem 5 of [17], since A has stable rank one, for each $t \in (0, \|x\|]$, there is a unitary $u_t \in \tilde{A}$ such that $u_t p_t = v p_t$, where $x = v|x|$ is the polar decomposition of x in A^{**}. Then

$$u_{t'} p_t = u_{t'} p_{t'} p_t = v p_{t'} p_t = v p_t = u_t p_t$$

for any $0 < t' < t < \|x\|$. In particular,

$$u_{t'} p_t u_{t'}^* = u_t p_t u_t^* = q_t.$$

□

Proposition 1.5. *Let A be a C^*-algebra with stable rank one and let $a, b \in A_+$ be two positive element. Then the following are equivalent.*

(1) $[p_a] \leq [p_b]$,
(2) $a \lessapprox b$,
(3) $a \lesssim b$.

Proof. From the definition, (2) implies (3). It is also known that (1) and (2) are equivalent. It remains to show (3) implies (1). To simplify notation, one may assume that A is unital, $0 \leq a, b \leq 1$. Suppose (3) holds. Let $\{\epsilon_n\}$ be a strictly decreasing sequence of positive numbers in $(0, 1]$ such that $\sum_{n=1}^\infty \epsilon_n \leq 1/2$.

By [18], there is a unitary $w_1 \in A$ such that

$$b_1 = w_1 f_{\epsilon_1/4}(a) w_1^* \leq w_1 f_{\epsilon_1/16}(a) w_1^* = \bar{b}_1 \in Her(b). \tag{e 1.1}$$

Note that $\bar{b}_1 b_1 = b_1$. Let $x_1 = w_1 (f_{\epsilon_1/4}(a))^{1/2}$. Then

$$x_1^* x_1 = f_{\epsilon_1/4}(a) \text{ and } x_1 x_1^* = b_1. \tag{e 1.2}$$

There is a unitary $w_2 \in A$ such that

$$w_2 w_1 f_{\epsilon_2/8}(a) w_1^* w_2^* = b_2 \in Her(b). \tag{e 1.3}$$

Denote $a_1 = w_2 w_1 f_{\epsilon_1/4}(a) w_1 w_2^*$. Note that $a_1 \in Her(b)$ and $a_1 b_2 = a_1$. Therefore

$$(b_2 - 1) w_2 w_1 |x_1| w_1^* = ((b_2 - 1) w_2 w_1 |x_1| w_1^* w_2^*) w_2 = 0. \tag{e 1.4}$$

In other words,

$$b_2 w_2 w_1 |x_1| w_1^* = w_2 w_1 |x_1| w_1^*. \tag{e 1.5}$$

Similarly,

$$w_2 w_1 |x_1| w_1^* \bar{b}_1 = w_2 w_1 |x_1| w_1^*. \tag{e 1.6}$$

Therefore $y_1 := w_2 w_1 |x_1| w_1^* \in Her(b)$. Moreover,

$$y_1^* y_1 = w_1 x_1^* x_1 w_1^* = x_1 x_1^* \text{ and } y_1 y_1^* = w_2 w_1 f_{\epsilon_1/4}(a) w_1^* w_2^*. \tag{e 1.7}$$

By applying 1.4, one obtains a unitary $z_1 \in \tilde{Her}(b)$ such that

$$z_1 e_{1/4}(|y_1|) = w_2 e_{1/4}(|y_1|) = w_2 e_{1/4}(|x_1^*|), \tag{e 1.8}$$

where $e_{1/4}(|y_1|)$ is the open spectral projection of $|y_1| = |x_1^*|$ associated with $(1/4, 1]$. Note that,

$$\begin{aligned} e_{1/4}(|x_1^*|) &= e_{1/4}(w_1 f_{\epsilon_1/4}(a) w_1^*) && \text{(e 1.9)} \\ &= w_1 e_{1/4}(f_{\epsilon_1/4}(a)) w_1^* && \text{(e 1.10)} \\ &= w_1 e_{\delta_1}(a) w_1^* && \text{(e 1.11)} \end{aligned}$$

where $e_{1/4}(f_{\epsilon_1/4}(a))$ is the open spectral projection of $f_{\epsilon_1/4}(a)$ associated with $(\epsilon_1/4, 1]$ and $e_{\delta_1}(a)$ is the open spectral projection of a associated with $(\delta_1, 1]$ for some $\delta_1 \in (\epsilon_1/4, 3\epsilon_1/8)$.

By (e 1.8) and (e 1.11),

$$\begin{aligned} z_1^* w_2 w_1 e_{\delta_1}(a) &= z_1^* (w_2 w_1 e_{\delta_1}(a) w_1^*) w_1 && \text{(e 1.12)} \\ &= z_1^* (z_1 e_{1/4}(|x_1^*|)) w_1 = e_{1/4}(|x_1^*|) w_1 && \text{(e 1.13)} \\ &= w_1 e_{\delta_1}(a) && \text{(e 1.14)} \end{aligned}$$

Define $u_1 = w_1$ and $u_2 = z_1^* w_2 u_1 = z_1^* w_2 w_1$, where one may view z_1 as a unitary in A. It follows, for any $x \in \overline{f_{\delta_1}(a) A}$, by applying (e 1.14), that

$$u_2 x = u_2 e_{\delta_1}(a) x = z_1^* w_2 w_1 e_{\delta_1}(a) = u_1 e_{\delta_1}(a) x = u_1 x \tag{e 1.15}$$

Note also that $u_2 y u_2^* \in Her(b)$ for all $y \in Her(f_{\epsilon_2/8}(a))$, and $f_{\epsilon_1}(a) \in Her(e_{\delta_1}(a))$.

By induction, for each n, one obtains a sequence of unitaries $u_n \in A$ such that

$$\begin{aligned} u_n y u_n^* &\in Her(b) \text{ for all } y \in Her(f_{\epsilon_n/8}(a)) \text{ and} && \text{(e 1.16)} \\ u_{n+1} x &= u_n x \text{ for all } x \in \overline{f_{\epsilon_n}(a) A} && \text{(e 1.17)} \end{aligned}$$

One then computes that

$$\lim_{n \to \infty} u_n x. \tag{e 1.18}$$

converges for every $x \in \overline{aA}$, which defines a unitary isomorphism U from $\overline{aA}$ into a Hilbert sub-module of $\overline{bA}$, which implies that $[p_a] \leq [p_b]$.

□

Remark 1.6. There will be some discussion of Hilbert modules in the last section. A countably generated Hilbert module may not have a countable dense set. Note that in Proposition 1.5, A is not assumed to be separable. The argument above can also be used to prove the following theorem which was proved in [8].

Theorem 1.7. *Let A be a C^*-algebra of stable rank one and let $a, b \in A_+$. Then the following are equivalent:*
(1) $[p_a] = [p_b]$;
(2) $\langle p_a \rangle = \langle p_b \rangle$;
(3) $\langle a \rangle = \langle b \rangle$.
In particular, $Co(A) = W(A)$ and $Co(A \otimes \mathcal{K}) = Cu(A)$.

2 Homotopy

It seems quite appropriate to begin with the following result of L. G. Brown ([3]).

Proposition 2.1. *Let A be a C^*-algebra and let p and q be two σ-compact open projections of A in A^{**}. Suppose that there is a norm continuous path $\{p(t) : t \in [0,1]\}$ of σ-compact open projections such that*

$$p(0) = p \text{ and } p(1) = q. \tag{e 2.19}$$

Then $[p] = [q]$.

Proof. Let $0 = t_0 < t_1 < t_2 < \cdots t_n = 1$ be a partition such that

$$\|p(t_i) - p(t_{i-1})\| < 1/2, \quad i = 1, 2, ..., n.$$

It follows from (the proof of) 3.2 of [3] that

$$[p(t_i)] = [p(t_{i-1})], \quad i = 1, 2, ..., n. \tag{e 2.20}$$

Thus $[p] = [q]$.

□

Definition 2.2. Let A be a C^*-algebra. An open projection $q \in A^{**}$ is said to be pre-compact, if there is a positive element $a \in A_+$ such that $qa = qa = q$. If p is another open projection and if there is $a \in Her(p)$ such that $qa = aq = q$, then one says that q *is compactly contained* in p.

Lemma 2.3. *Let A be a C^*-algebra and let $a \in A_+$. Suppose that $q \in A^{**}$ is a projection for which $qa = aq = q$. Then, $\chi_{(1,\|a\|]}(a)q = q$.*

Proof. Let $s > 1$. Denote by $p_{(s,\|a\|]}$ the spectral projection of a in A^{**} corresponding to the interval $(s, \|a\|]$. Since $qa = aq$, q commutes with $p_{(s,\|a\|]}$. In particular, $qp_{(s,\|a\|]}$ is a projection. However, $qp_{(s,\|a\|]} = 0$. Otherwise

$$1 < s \leq \|qap_{(s,\|a\|]}\| \leq \|aq\| = \|q\| = 1. \tag{e 2.21}$$

It follows that

$$qp_{(1,\|a\|]} = 0. \tag{e 2.22}$$

Let $0 < r < 1$ and $p_{[0,r]} = \chi_{[0,r]}(a)$ be the spectral projection corresponding to the interval $[0, r]$. The assumption that $qa = aq = q$ implies that

$$p_{[0,r]}q = qp_{[0,r]}. \tag{e 2.23}$$

It follows that

$$q = aq \le rp_{[0,r]}q + ap_{(r,\|a\|]}q \le rp_{[0,r]}q + p_{(r,1]}q \le p_{[0,r]}q + p_{(r,1]}q = q. \tag{e 2.24}$$

It follows that

$$rp_{[0,r]}q = p_{[0,r]}q. \tag{e 2.25}$$

Therefore

$$p_{[0,r]}q = 0. \tag{e 2.26}$$

Since this holds for each $r \in (0,1)$, one concludes that

$$q = p_a q = \chi_{\{1\}}(a)q = q. \tag{e 2.27}$$

□

Definition 2.4. Let p be a σ-compact open projection of A which is not in A. Let $a \in Her(p)$ be a strictly positive element. Then 0 must be a limit point of $\mathrm{sp}(a)$. Let $t_n \in (0, \|a\|]$ be such that $t_n \searrow 0$. Let $p_{K,n}$ be the open spectral projection corresponding to $(t_n, \|a\|]$. Then $f_{t_n/2}(a) \ge p_{K,n}$, So $p_{K,n}$ is a sub-pre-compact open projection of p. Note that $\{p_{K,n} : n = 1, 2, ...\}$ is increasing and

$$\lim_{n\to\infty} p_{K,n} = p$$

in the strong operator topology in A^{**}. Such a sequence $\{p_{K,n}\}$ is called a *pre-compact support* of p.

In the proof of 2.5, 2.6 and 2.7, the result of L. G. Brown and G. K. Pedersen (3.6 of [5]) that every hereditary C^*-subalgebra of a C^*-algebra of stable rank one has stable rank one will be used without repeating this reference.

Lemma 2.5. *Let A be a simple C^*-algebra of stable rank one and let $x \in A$. Suppose that $x = v|x|$ is the polar decomposition of x in A^{**}. Suppose also that 0 is not an isolated point in* $\mathrm{sp}(x)$. *Then, for any $\delta > 0$, there is a unitary $u \in \tilde{A}$ with $[u] = 0$ in $K_1(A)$ such that*

$$up_t = vp_t \ \textit{for all} \ t \in [\delta, \|x\|], \tag{e 2.28}$$

where p_t is the spectral projection of $|x|$ corresponding to $(t, \|x\|]$.

Proof. It follows from 1.4 that there is a unitary $u_\delta \in \tilde{A}$ such that

$$u_\delta p_t = vp_t \ \text{for all} \ t \in [\delta, \|x\|]. \tag{e 2.29}$$

Since 0 is not an isolated point in $\mathrm{sp}(x)$, there are $0 < t' < t'' < \delta$ such that $p_{(t',t'')} \ne 0$, where $p_{(t',t'')}$ is the spectral projection of $|x|$ corresponding to (t', t''). Note that $p_{(t',t'')}$ is an open projection of A. Let $B = Her(p_{(t',t'')})$. Then B has stable rank one. Since A is also simple, the map $K_1(B) \to K_1(A)$ induced by the inclusion is an isomorphism. Therefore there is a unitary $v \in \widetilde{B}$ such that $[v] = [u_\delta^*]$ in $K_1(A)$. One may write $v = z + \lambda$, where $z \in B$ and $\lambda \in \mathbb{C}$. Let $\pi : \widetilde{B} \to \mathbb{C}$ be the quotient map. Then $\pi(v) = \lambda$. It follows that $|\lambda| = 1$. Put $v_1 = \overline{\lambda}v = \overline{\lambda}z + 1$. Note that

$$zp_t = 0 \ \text{for all} \ t \in (\delta, \|x\|] \ \text{and} \ [v_1] = [v] = [u_\delta^*] \ \text{in} \ K_1(A). \tag{e 2.30}$$

One may view v_1 as a unitary in $\tilde{A}$. Now set $u = u_\delta v_1$. Then,

$$\begin{aligned} up_t &= u_\delta v_1 p_t = u_\delta(\overline{\lambda} z + 1)p_t && \text{(e 2.31)} \\ &= u_\delta p_t = p_t && \text{(e 2.32)} \end{aligned}$$

for all $t \in [\delta, \|x\|]$.

□

Theorem 2.6. *Let A be a simple C^*-algebra with stable rank one. Suppose that p and q are two σ-compact open projections of A such that $[p] = [q]$. Then, there is a precompact support $\{p_{K,n}\}$ of p, and there is a continuous path of unitaries $\{w(t) : t \in [0,1)\} \subset \tilde{A}$ satisfying the following: $w(0) = 1$, for any n, there is $t_n \in (0,1)$ such that*

$$w(t)p_{K,n}w(t)^* = w(t_n)p_{K,n}w(t_n)^* \text{ for all } t \in [t_n, 1) \tag{e 2.33}$$

and $\{w(t_n)^* p_{K,n} w(t_n)\}$ is a precompact support of q. Moreover,

$$w(t)p_{K,n} = \lambda(t)w(t_n)p_{K,n} \tag{e 2.34}$$

for some $\lambda(t) \in \mathbb{C}$ if $t \in [t_n, 1)$.

Proof. Suppose that $[p] = [q]$. If p is a projection in A, so is q. Then the result follows from a theorem of L. G. Brown (Theorem 1 of [4]).

So, one now assumes that neither p nor q are projections in A. Let $a \in Her(p)$ be a strictly positive element. Let $p_{K,n}$ be the spectral projection of a associated with $(1/2^{n+1}, \|a\|]$. Then $\{p_{K,n}\}$ is a precompact support for p. Suppose that $w \in A^{**}$ such that

$$w^*w = p,\ ww^* = q \text{ and } wbw^* \in Her(q) \text{ for all } b \in Her(p). \tag{e 2.35}$$

Put $x = wa^{1/2}$. Then $xx^* = waw^*$ is a strictly positive element of $Her(q)$.

Put $s_1 = 1/\sqrt{2}$, $s_n = 1/2^{n-1}$, $n = 1, 2, ...$ Since one assumes that p is not a projection in A, 0 is a limit point of $\mathrm{sp}(a^{1/2})$. Let p_{s_n} be the open spectral projection of $|x| = a^{1/2}$ associated with $(s_n, \|x\|]$. Then $p_{s_n} = p_{K,n}$, $n = 1, 2, ...$ Let $t_n = s_n - s_n/16^n$ and let p_{t_n} be the open spectral projection of $|x|$ associated with $(t_n, \|x\|]$, $n = 1, 2, ...$

It follows from 2.5 (see also 1.4) that there is a unitary $u_n \in \tilde{A}$ with $[u_n] = 0$ in $K_1(A)$ such that

$$u_m p_{t_n} u_m^* = q_{t_n} \text{ and } u_m p_t = u_n p_t, \text{if } m \ge n, t \ge t_n, n, m = 1, 2, ..., \tag{e 2.36}$$

where q_{t_n} is the open spectral projection of $|x^*|$ associated with $(t_n, \|x^*\|]$.

Denote by q_{s_n} the spectral projection of $|x^*|$ associated with $(s_n, \|x^*\|]$, $n = 1, 2, ...$ Since $[u_1] = 0$ in $K_1(A)$ and A has stable rank one, by a result of Rieffel ([19]), $u_1 \in U_0(\tilde{A})$. Therefore there is a continuous path of unitaries $\{w(t) : t \in [0, t_1]\} \subset \tilde{A}$ $(0 < t_1 < 1)$ such that

$$w(0) = 1, \text{ and } w(t_1) = u_1. \tag{e 2.37}$$

On also has that

$$\begin{aligned} u_2 u_1^* q_{t_1} &= u_2 u_1^*(u_1 p_{t_1} u_1^*) && \text{(e 2.38)} \\ &= u_2 p_{t_1} u_1^* = u_1 p_{t_1} u_1^* = q_{t_1} && \text{(e 2.39)} \\ &= u_1 p_{t_1} u_1^* = u_2 p_{t_1} u_1^* = u_2 p_{t_1} u_2^*(u_2 u_1^*) = q_{t_1}(u_2 u_1^*). && \text{(e 2.40)} \end{aligned}$$

Moreover,

$$u_2u_1^*q_{t_2}(u_2u_1^*)^* = u_2u_1^*(u_1p_{t_2}u_1^*)u_1u_2^* = u_2p_{t_2}u_2^* = q_{t_2}. \tag{e 2.41}$$

Let $e_1 = p_{(0,(t_1+s_1)/2)}$ the spectral projection of $|x|$ corresponding to $(0,(t_1+s_1)/2)$ and let $C = Her(e_1)$. By (e 2.39) and (e 2.40), one may view $u_2u_1^*$ as a unitary in $\widetilde{C}$. Since C has stable rank one and $[u_2u_1^*] = 0$ in $K_1(A)$, one obtains a continuous path of unitaries $\{W(t) : [t_1,t_2]\} \subset \widetilde{C}$ $(t_1 < t_2 < 1)$ such that

$$W(t_1) = 1 \text{ and } W(t_2) = u_2u_1^*. \tag{e 2.42}$$

Note that

$$q_{s_1}e_1 = e_1q_{s_1} = 0. \tag{e 2.43}$$

$W(t)$ may be viewed as unitaries in $\tilde{A}$. Moreover, by (e 2.43),

$$W(t)q_{s_1} = \lambda(t)q_{s_1} = q_{s_1}W(t) \tag{e 2.44}$$

for some $\lambda(t) \in \mathbb{C}$ for all $t \in [t_1,t_2]$. Now extend $w(t)$ from a continuous path from $[0,t_1]$ to a continuous path from $[0,t_2]$ by defining

$$w(t) = W(t)w(t_1) \text{ for all } t \in [t_1,t_2]. \tag{e 2.45}$$

Note that

$$w(t_2)p_{s_2}w(t_2)^* = q_{s_2} \text{ and } w(t)p_{s_1}w(t)^* = q_{s_1} \tag{e 2.46}$$

for all $t \in [t_1,t_2]$. Moreover, by (e 2.44),

$$\begin{aligned} w(t)p_{s_1} &= W(t)w(t_1)p_{s_1} = W(t)w(t_1)p_{s_1}w(t_1)^*w(t_1) && \text{(e 2.47)}\\ &= W(t)q_{s_1}w(t_1) = \lambda(t)q_{s_1}w(t_1) && \text{(e 2.48)}\\ &= \lambda(t)w(t_1)p_{s_1}w(t)^*w(t_1) = \lambda(t)w(t_1)p_{s_1}. && \text{(e 2.49)} \end{aligned}$$

Furthermore,

$$w(t_2) = u_2u_1^*w(t_1) = u_2u_1^*u_1 = u_2. \tag{e 2.50}$$

One also has that

$$\begin{aligned} u_3w(t_2)^*q_{t_2} &= u_3u_2^*q_{t_2}u_2u_2^* = u_3p_{t_2}u_2^* = u_2p_{t_2}u_2^* = q_{t_2} && \text{(e 2.51)}\\ &= u_2p_{t_2}u_2^* = u_3p_{t_3}u_3^*(u_3u_2^*) = q_{t_2}(u_3w(t_2)^*), \text{ and} && \text{(e 2.52)} \end{aligned}$$

$$\begin{aligned} u_3w(t_2)^*q_{t_3}w(t)u_3^* &= u_3w(t_2)^*(w(t_2)p_{t_3}w(t_2)^*)w(t_2)u_3^* && \text{(e 2.53)}\\ &= u_3p_{t_3}u_3^* = q_{t_3}. && \text{(e 2.54)} \end{aligned}$$

Therefore, by induction, one obtains a continuous path of unitaries $\{w(t) : t \in [0,1)\}$ of $\tilde{A}$ such that

$$w(0) = 1, w(t_n)p_{s_n}w(t_n)^* = q_{s_n} \text{ and} \tag{e 2.55}$$

$$w(t)p_{s_n}w(t)^* = q_{s_n} \text{ for all } t \in [t_n,1). \tag{e 2.56}$$

Moreover,

$$w(t)p_{s_n} = \lambda(t)w(t_n)p_{s_n} \text{ for all } t \in [t_n,1) \tag{e 2.57}$$

for some $\lambda(t) \in \mathbb{C}$.

□

Corollary 2.7. *Let A be a simple C^*-algebra with stable rank one and let a, $b \in A_+$. Suppose that $[p_a] \le [p_b]$. Then, for any $c \in Her(a)_+$ which is compactly contained in p_a, there exists a continuous path of unitaries $\{w(t) : t \in [0,1]\}$ such that $w(0) = 1$ and $w(1)^* p_c w(1)$ is compactly contained in p_b.*

Proof. Suppose that $c \in Her(p_a)_+$ which is compactly contained in $Her(p_a)$ in the sense that there is $d \in Her(p_a)_+$ such that $cd = c$. Then, for any $\epsilon > 0$, there is an integer $n \ge 1$ such that

$$\|f_{1/n}(a)d - d\| < \epsilon/2. \tag{e 2.58}$$

It follows from [18] that

$$f_\epsilon(d) \lessapprox f_{1/n}(a). \tag{e 2.59}$$

Since A has stable rank one, then there exists $v \in \tilde{A}$ such that $v^* f_\epsilon(d) v \le f_{1/n}(a)$. By 2.3,

$$c f_\epsilon(d) = c. \tag{e 2.60}$$

Let p_c be the range projection of c in A^{**}. Then, $v^* p_c v \le p_{K,n}$. Since A is simple and has stable rank one, there is $v_0 \in \widetilde{Her(p_c)}$ such that

$$[v_0] = [v^*] \text{ in } K_1(A). \tag{e 2.61}$$

One may also view v_0 as a unitary in $\tilde{A}$. There is a continuous path of unitaries $\{w_0(t) : t \in [0,1]\} \subset \tilde{A}$ such that

$$w_0(0) = 1, w_0(1) = v_0 v. \tag{e 2.62}$$

Then

$$w_0(1)^* p_c w_0(1) = v^* p_c v \le p_{K,n}. \tag{e 2.63}$$

Now the lemma follows from 2.6.

□

Proposition 2.8. *The converse of Theorem 2.6 also holds in the following sense. Let A be a C^*-algebra and let p and q be two σ-compact open projections of A. Suppose that there is a continuous path of unitaries $\{w(t) : t \in [0,1)\} \subset M(A)$ such that, $w(0) = 1$, for any $n \ge 1$, there is $t_n \in (0,1)$ such that*

$$w(t) p_{K,n} w(t)^* = w(t_n) p_{K,n} w(t_n)^* \text{ for all } t \in [t_n, 1), \tag{e 2.64}$$

where $\{p_{K,n}\}$ is a precompact support for p and $\{w(t_n)^* p_{K,n} w(t_n)\}$ is a precompact support for q. Moreover,

$$w(t) p_{K,n} = \lambda(t) w(t_n) p_{K,n} \text{ for all } t \in [t_n, 1) \tag{e 2.65}$$

for some $\lambda(t) \in \mathbb{C}$.

Then $[p] = [q]$.

Proof. One may assume that $t_{n+1} > t_n$, $n = 1, 2, \ldots$. Suppose that $a \in Her(p)$ is a strictly positive element and suppose $s_n \in (0, \|a\|]$ such that $s_n \searrow 0$ such that $p_{K,n}$ is the spectral projection of a corresponding to $(s_n, \|a\|]$, $n = 1, 2, \ldots$. One defines, with $p_{K,0} = 0$,

$$v = \sum_{n=1}^{\infty} w(t_n)(p_{K,n} - p_{K,n-1}). \tag{e 2.66}$$

One checks that, for $b \in Her(p)$,

$$\lim_{n\to\infty} \| \sum_{k=n}^{n+m} w(t_n)(p_{K,n} - p_{K,n-1})b\| = 0. \tag{e 2.67}$$

It follows that $v \in A^{**}$. One also checks that

$$\sum_{n=1}^{m} w(t_n)(p_{K,n} - p_{K,n-1})w(t_m)^* \tag{e 2.68}$$

$$= \sum_{n=1}^{m} \overline{\lambda(t_m)} w(t_m)(p_{K,m} - p_{K,n-1})w(t_m)^* \tag{e 2.69}$$

$$= \overline{\lambda(t_m)} w(t_m)p_{K,m}w(t_m)^* = \overline{\lambda(t_m)} q_{K,m}. \tag{e 2.70}$$

Let $a_n = f_{t_n}(a)$, $n = 1, 2, ...$ Then, if $m > n + 1$,

$$va_nv^* = \overline{\lambda(t_m)} q_{K,m} w(t_m)a_n w(t_m)^* q_{K,m} \overline{\lambda(t_m)} \tag{e 2.71}$$

$$= w(t_m)a_n w(t_m)^* \in Her(q). \tag{e 2.72}$$

Let $\epsilon > 0$ and $b \in Her(p)$. There is $n \geq 1$ such that

$$\|b - a_m b a_m\| < \epsilon \text{ for all } m \geq n. \tag{e 2.73}$$

Then,

$$\|vbv^* - va_m b a_m v^*\| < \epsilon. \tag{e 2.74}$$

But, by (e 2.72), $va_m b a_m v^* \in Her(p)$. This implies that $vbv^* \in Her(p)$. Furthermore,

$$v^*pv = q. \tag{e 2.75}$$

□

Remark 2.9. If A is a unital C^*-algebra and $p, q \in A$ are two projections which are homotopy, i.e., there is a projection $P \in C([0, 1], A)$ such that $P(0) = p$ and $P(1) = q$. Then (see, for example, Lemma 2.6.6 of [16]), there is a unitary $U \in C([0, 1], A)$ such that $U(0) = 1$ and $U(t)^*pU(t) = p(t)$ for all $t \in [0, 1]$. In 2.6 and in 2.8, for each $n \geq 1$, there is a continuous path

$$\{p(t) = w(t)^*p_{K,n}w(t) : t \in [0, 1)\}$$

of open projections such that

$$p(0) = p_{K,n} \text{ and } p(t) = q_{K,n} \text{ for all } t \in [t_n, 1).$$

In particular, there is a unitary, namely $u(t_n) \in \tilde{A}$ or in $M(A)$ in 2.8), such that

$$u(t_n)^*p_{K,n}u(t_n) = q_{K,n}, \quad n = 1, 2, ... \tag{e 2.76}$$

In general, however, if $[p] = [q]$ in the sense of Cuntz, there may not be any unitary path $\{w(t) : t \in [0, 1]\}$ for which $w(0)^*pw(0) = p$ and $w(1)^*pw(1) = q$, as one can see from the following.

Proposition 2.10. *Let A be a non-unital and σ-unital non-elementary simple C^*-algebra with (SP). Then there are two σ-compact open projections p and q of A such that $[p] = [q]$ but there are no unitary $u \in M(A)$ such that $u^*pu = q$.*

Proof. Let $a \in A_+$ be a strictly positive element. Then 0 is a limit point of $\mathrm{sp}(a)$. Thus A admits an approximate identity $\{e_n\}$ such that $e_{n+1}e_n = e_n$, $n = 1, 2, ...$ One may further assume, without loss of generality, by passing to a subsequence if necessary, there are nonzero positive element $b_n \in Her(e_{2(n+1)} - e_{2n})$ with $\|b_n\| = 1$, $n = 1, 2, ...$ In particular,

$$b_i b_j = 0 \text{ if } i \neq j. \tag{e 2.77}$$

On the other hand, since $\overline{e_1 A e_1}$ is a non-elementary simple C^*-algebra, by a result of Akemann and Shultz [1], there are mutually orthogonal non-zero positive elements $c_1, c_2, ..., c_n, ...$ in $\overline{e_1 A e_1}$. Since A has (SP), there are non-zero projections $d'_n \in Her(c_n)$, $n = 1, 2, ...$

By a result of Cuntz (see (2) of Lemma 3.5.6 of [16], for example), there are partial isometries $x_1, x_2, ..., x_n, ... \in A$ such that

$$x_i^* x_i \in Her(c_i) \text{ and } x_i x_i^* \in Her(b_i), \tag{e 2.78}$$

where $x_i^* x_i$ and $x_i x_i^*$ are non-zero projections, $i = 1, 2, ...$

Put $d_n = x_n x_n^*$ and $f_n = x_n^* x_n$, $n = 1, 2, ...$ Define

$$b = \sum_{n=1}^{\infty} \frac{y_n}{n^2} \text{ and } c = \sum_{n=1}^{\infty} \frac{z_n}{n^2}. \tag{e 2.79}$$

Then $b, c \in A$. Define $x = \sum_{n=1}^{n} \frac{x_n}{n}$. Then

$$x^* x = b \text{ and } x x^* = c. \tag{e 2.80}$$

Let $p = p_c$, the range projection of c in A^{**} and let $q = p_b$, the range projection of b in A^{**}. Then, by (e 2.80), $[p] = [q]$. Moreover $ce_2 = c$. So c is compact. Furthermore, since $f_n \leq (e_{2(n+1)} - e_{2n})$, $n = 1, 2, ...,$

$$q = \sum_{n=1}^{\infty} f_n, \tag{e 2.81}$$

where the sum converges in the strict topology. It follows that $q \in M(A)$.

Now suppose that there were a unitary $u \in M(A)$ such that $u^* q u = p$. Therefore $u^* q u \in M(A)$. However $p \notin M(A)$. Otherwise $pe_2 = p$ implies that $p \in A$. But $p \notin A$. So there is no unitary $u \in M(A)$ for which $u^* q u = p$.

□

Remark 2.11. In the proof of 2.10, one notes that p is precompact and is compactly contained in p_{e_2}. However, q is not precompact and is not compactly contained in any σ-compact open projection of A. In fact, if $q \leq a$ for some $a \in A_+$, then $q \in A$ since $q \in M(A)$. One concludes that precompactness is not invariant under the Cuntz relation.

Finally, to end this section, one has the following:

Proposition 2.12. *Let A and B be two separable C^*-algebras, and let $\varphi_0, \varphi_1 : A \to B$ be two homomorphisms. Suppose that there is a homomorphism $H : A \to C([0,1], B)$ such that $\pi_0 \circ H = \varphi_0$ and $\pi_1 \circ H = \varphi_1$, where $\pi_t : C([0,1], B) \to B$ is the point-evaluation at the point $t \in [0,1]$. Suppose also that H extends to a (sequentially) normal homomorphism $H' : A^{**} \to C([0,1], B^{**})$ in the sense that if $\{a_n\} \subset A_{s.a}$ is a increasing bounded sequence with upper bound $x \in A^{**}$, then $\{H(a_n)\}$ has the upper bound $H'(x)$. Then φ_0 and φ_1 induce the same homomorphism on the Cuntz semigroups $W(A)$ and $Co(A)$.*

Proof. Let p be an open projection of A. Since A is separable, p is σ-compact. Let $a \in A_+$ such that p is the range projection of A. Then $\{a^{1/n}\}$ is increasing and has the upper bound p.

Put $p(t) = \pi_t \circ H'(p)$, $t \in [0,1]$. It is a norm continuous path of σ-compact open projections. It follows from 2.1 that $[p(0)] = [p(1)]$. In other words,

$$[\varphi_0(p)] = [\varphi_1(p)]. \tag{e 2.82}$$

□

3 Hilbert Modules

From the definition (see 1.1), two σ-compact open projections p and q of a C^*-algebra A are Cuntz equivalent if and only if the corresponding Hilbert A-modules are isomorphic as Hilbert A-modules. When A has stable rank one, by [8] (see also 1.7), two positive elements a and b in A are Cuntz equivalent if and only if the associated Hilbert A-modules are isomorphic as Hilbert A-modules. A question was mentioned in [8] (see line 27 of page 187 of [8]) whether a and b are Cuntz equivalent if the Banach A-modules $\overline{aA}$ and $\overline{bA}$ are isomorphic as Banach modules. This question will be answered by a result of L. G. Brown below (3.1 and 3.2). Recently, related to the Cuntz semigroups, projective Hilbert modules also attract some attention (see [6]). In this section, these two issues will be discussed. We begin with the following definition.

Let A be a C^*-algebra. For an integer $n \geq 1$, denote by $A^{(n)}$ the Hilbert A-module of orthogonal direct sum of n copies of A. If $x = (a_1, a_2, ..., a_n), y = (b_1, b_2, ..., b_n)$, then

$$< x, y >= \sum_{i=1}^{n} a_n^* b_n.$$

Denote by H_A the standard countably generated Hilbert (right) A-module

$$H_A = \{\{a_n\} : \sum_{n=1}^{k} a_n^* a_n \text{ converges in norm}\},$$

where the inner product is defined by

$$< \{a_n\}, \{b_n\} >= \sum_{n=1}^{\infty} a_n^* b_n.$$

Let H be a Hilbert A-module. denote by $H^\sharp$ the set of all bounded A-module maps from H to A. If H_1, H_2 are Hilbert A-modules, denote by $B(H_1, H_2)$ the space of all bounded module maps from H_1 and H_2. If $T \in B(H_1, H_2)$, denote by $T^* : H_2 \to H_1^\sharp$ the bounded module maps defined by

$$T^*(y)(x) =< Tx, y > \text{ for all } x \in H_1 \text{ and } y \in H_2.$$

If $T^* \in B(H_2, H_1)$, one says that T has an adjoint T^*. Denote by $L(H_1, H_2)$ the set of all bounded A-module maps in $B(H_1, H_2)$ with adjoints. Let H be a Hilbert A-module. In what follows, denote $B(H) = B(H, H)$ and $L(H) = L(H, H)$. $B(H)$ is a Banach algebra and $L(H)$ is a C^*-algebra.

Denote by $F(H)$ the linear span of those module maps with the form $\xi < \zeta, - >$, where $\xi, \zeta \in H$. Denote by $K(H)$ the closure of $F(H)$. $K(H)$ is a C^*-algebra. It follows from a result of Kasparov ([11]) that $L(H) = M(K(H))$, the multiplier algebra of $K(H)$, and, by [14], $B(H) = LM(K(H))$, the left multiplier algebra of $K(H)$.

Two Hilbert A-modules are said to be unitarily equivalent, or isomorphic, if there is an invertible map $U \in B(H_1, H_2)$ such that

$$< U(x_1), U(x_2) >=< x_1, x_2 > \text{ for all } x_1, x_2 \in H_1.$$

The following result of L. G. Brown becomes quite useful and answers the question in [8] mentioned above.

Theorem 3.1. (Theorem 3.2 of [3] and Theorem 2.2 of [15]) *Let H_1 and H_2 be two countably generated Hilbert modules over a C^*-algebra A. Suppose that there is $T \in B(H_1, H_2)$ which is one-to-one and has dense range. Then H_1 and H_2 are unitarily equivalent.*

Remark 3.2. However, it is also worth to note that the above statement fails when H_1 and H_2 are not countably generated. See Example 2.3 of [15].

As a consequence, one has the following.

Proposition 3.3. *Let A be a C^*-algebra and let $a, b \in A_+$. Suppose that $H_1 = \overline{aA}$ and $H_2 = \overline{bA}$. Then $[p_a] \leq [p_b]$ (or equivalently, $a \lessapprox b$) if and only if there is $T \in B(H_2, H_1)$ whose range is dense in H_1.*

Proof. Suppose that $[p_a] \leq [p_b]$, i.e., there is a partial isometry $v \in A^{**}$ such that

$$v^* p_a v \leq p_b \text{ and } v^* x v \in Her(b) \text{ for all } x \in Her(a). \tag{e 3.83}$$

Thus $v^* H_1 \subset H_2$. Put $H_3 = \overline{v^* H_1}$ and $c = v^* a v$. Then $c \in K(H_3)$. It follows from Lemma 2.13 of [15] that one may view $K(H_3)$ as a hereditary C^*-subalgebra of $K(H_1)$. Thus $T = vc$ defines a bounded module map in $B(H_2, H_1)$. Note $T = av$ and $\overline{vH_2} = H_1$. It follows that T has the dense range.

Now one assumes that there is $T \in B(H_2, H_1)$ whose range is dense in H_1. One may identify T with an element in $LM(Her(b), Her(a))$. Let $x = (Tb)^* Tb$. Then $x \in Her(b)$. Let $H_4 = \overline{xA}$. Then T is one-to-one on H_4 and has dense range. It follows from 3.1 that H_4 and H_1 are unitarily equivalent which provides a partial isometry $v \in A^{**}$ such that

$$v\overline{aA} = \overline{xA} \text{ and , for } \xi \in \overline{aA}, \ v\xi = 0 \text{ if and only if } \xi = 0 \tag{e 3.84}$$

Let $r = p_x$. Then

$$v p_a v^* = r \leq p_b \text{ and } v \xi v^* \in Her(x) \subset Her(b) \text{ for all } \xi \in Her(a). \tag{e 3.85}$$

□

Now we turn to the projectivity of Hilbert modules.

Theorem 3.4. *Let A be a C^*-algebra. Then every Hilbert A module is projective (with bounded module maps with adjoints as morphisms) in the following sense: Let H be a Hilbert A-module.*

(1) *Suppose that H_1 is another Hilbert A-module and suppose that $\varphi \in L(H_1, H)$ is a surjective. Then there is $\psi \in L(H, H_1)$ such that*

$$\varphi \circ \psi = \mathrm{id}_H; \tag{e 3.86}$$

(2) *Suppose that H_2 and H_3 are Hilbert modules and suppose that $\varphi_1 \in L(H_2, H_3)$ is surjective. Suppose also that $\varphi_2 \in L(H, H_3)$. Then there exists $\psi \in L(H, H_2)$ such that*

$$\varphi_1 \circ \psi = \varphi_2. \tag{e 3.87}$$

Proof. For (1), one first notes that φ has closed range. Define $T : H_1 \oplus H \to H_1 \oplus H$ by $T(h_1 \oplus h) = 0 \oplus T(h_1)$ for $h_1 \in H_1$ and $h \in H$. Then $T \in L(H_1 \oplus H) = M(K(H_1 \oplus H))$. It follows from Lemma 2.4 of [15] that

$$H_1 \oplus H = \ker T \oplus |T|(H_1 \oplus H). \tag{e 3.88}$$

Let $T = V|T|$ be the polar decomposition in $(K(H_1 \oplus H))^{**}$. Note that the proof of Lemma 2.4 of [15] shows that 0 is an isolated point of $|T|$ or $|T|$ is invertible. So the same holds for $(TT^*)^{1/2}$. Let $S = (TT^*)^{-1}$, where the inverse is taken in the hereditary C^*-subalgebra $L(H) \subset L(H_1 \oplus H)$. Since T is surjective,

$$|TT^*|H = H. \tag{e 3.89}$$

Moreover,

$$L_1 = V^*(TT^*)^{-1/2} = V^*(TT^*)^{1/2}S \in L(H_1 \oplus H). \tag{e 3.90}$$

One then checks that

$$TL_1 = V|T|V^*(TT^*)^{-1/2} = P, \tag{e 3.91}$$

where P is the range projection of $(TT^*)^{1/2}$ which gives the identity of H. One then defines ψ by L_1. Thus $\varphi \circ \psi = \mathrm{id}_H$.

For (2), one applies (1). Since φ_1 is surjective, by (2), there is $\varphi_3 \in L(H_3, H_2)$ such that

$$\varphi_1 \varphi_3 = \mathrm{id}_{H_3}. \tag{e 3.92}$$

Define $\psi = \varphi_1 \circ \psi_3 \circ \varphi_2$.

□

Remark 3.5. A discussion about injective Hilbert modules can be found in [15]. It was shown that, for example, a Hilbert A-module H is injective (with bounded module maps with adjoints as morphisms) if and only if it is orthogonally complementary (Theorem 2.14 of [15]). For a full countably generated Hilbert module, it is injective (with bounded morphisms with adjoints as morphisms) if and only if $L(H) = B(H)$ (see 2.9 and 2.19 of [15]).

Let A be a C^*-algebra. One may consider the category of Hilbert A-modules with bounded A-module maps as morphisms. A discussion on the question which Hilbert A-modules are injective in this category was given in [15]. It seems that question which Hilbert A-modules are projective in this category is much more difficult. Consider a Hilbert A-module $H = \xi A$ which is singly algebraically generated. Let H_1 be another Hilbert A-module and $T \in B(H_1, H)$ is surjective. Suppose that $x \in H_1$ such that $T(x) = \xi$. It would be most natural to define $S : H \to H_1$ by $S(\xi) = x$ which gives $TS(y) = y$ for all $y \in H$. The trouble is that it is not clear why S should be bounded.

Noticing the difference between algebraically projective A-modules and projective Hilbert A-modules (with bounded module maps as morphisms), the following two propositions may not seem entirely trivial. The first one is certainly known. After this note was first posted, Leonel Robert informed the author that, using Proposition 3.4 above, he has a proof that the converse of the following also holds, i.e., if H is algebraically finitely generated, then $K(H)$ has an identity.

Proposition 3.6. *Let A be a C^*-algebra and let H be a Hilbert A-module. Suppose that $1_H \in K(H)$. Then H is algebraically finitely generated.*

Proof. Let $F(H)$ be the linear span of rank one module maps of the form $\xi < \zeta, - >$ for $\xi, \zeta \in H$. Then $F(H)$ is dense in $K(H)$. There is $T \in F(H)$ such that

$$\|1_H - T\| < 1/4, \tag{e 3.93}$$

One may assume that $\|T\| \le 1$. Thus

$$\|1_H - T^*T\| < 1/2. \tag{e 3.94}$$

It follows that $0 \le T^*T \le 1_H$ and T^*T is invertible. Note that $T^*T \in F(H)$. Therefore there are $\xi_1, \xi_2, ..., \xi_n, \zeta_1, \zeta_2, ..., \zeta_n \in H$ such that

$$T^*T(\xi) = \sum_{j=1}^{n} \xi_j < \zeta_j, \xi > \quad \text{for all} \quad \xi \in H. \tag{e 3.95}$$

But $T^*TH = H$. This implies that $\sum_{j=1}^{n} \xi_j A = H$.

□

Proposition 3.7. *Let A be a C^*-algebra and let H be a Hilbert A-module for which $K(H)$ has an identity. Then H is projective Hilbert A-module (with bounded module maps as morphisms).*

Proof. One first assumes that A has an identity. From 3.6, H is finitely generated. Therefore, a theorem of Kasparov shows that $H = PH_A$ for some projection $P \in L(H_A)$. The fact that $1_H \in K(H)$ implies that $P \in K(H_A)$. Therefore there is an integer $N \ge 1$ and a projection $P_1 \in M_N(A)$ such that PH is unitarily equivalent to $P_1 H_A$. In other words, one may assume that H is a direct summand of $A^{(N)}$. Suppose that H_1 and H_2 are two Hilbert A-modules and suppose that $S \in B(H_1, H_2)$ is surjective and suppose that $\varphi : H \to H_2$ is a bounded module map. Since H is a direct summand of $A^{(N)}$, there is a partial isometry $V \in L(H, A^{(N)})$ such that $P_1 V = \mathrm{id}_H$. Let $T = \varphi \circ P_1$. Denote by e_i the vector in the ith copy of A given by 1_A. Choose $g_1, g_2, ..., g_n \in H_1$ such that $Sg_i = Te_i$, $i = 1, 2, ..., n$. Define $L : K(A^{(N)} \oplus H_1)$ by

$$L(h \oplus h_1) = \sum_{j=1}^{N} g_i < e_i, h > \quad \text{for all} \quad h \in H \quad \text{and} \quad h_1 \in H_1. \tag{e 3.96}$$

Define $L_1 = L|_H$. For $h = \sum_{j=1}^{N} e_i a_i$, where $a_i \in A$, one has

$$\begin{aligned} SL_1(h) &= S(\sum_{j=1}^{N} g_i < e_i, h >) = \sum_{j=1}^{N} Sg_i < e_i, h > && \text{(e 3.97)} \\ &= \sum_{j=1}^{N} Te_i < e_i, e_i > a_i = \sum_{j=1}^{N} Te_i a_i && \text{(e 3.98)} \\ &= T(h). && \text{(e 3.99)} \end{aligned}$$

Define $L_2 \in B(H, H_1)$ by $L_2 = L_1 \circ V$. Then

$$SL_2 = SL_1 \circ V = T \circ V = \varphi \circ P_1 \circ V = \varphi. \tag{e 3.100}$$

Moreover, if $S_1 \in B(H_1, H)$ is a surjective map, consider the following diagram:

$$\begin{array}{ccccc} & & H & & \\ & & \downarrow_{\mathrm{id}_H} & & \\ H_1 & \to_{S_1} & H & \to 0 & \end{array}$$

From (ii), there is a bounded module map $L : H \to H_1$ such that

$$S_1 L = \mathrm{id}_H. \tag{e 3.101}$$

For general case, one may consider H as a Hilbert $\tilde{A}$-module.

□

Remark 3.8. The fact that $< e_i, e_i >= 1_A$ is crucial in the proof. It should be noted that, when A is not unital, the above argument does not imply that $A^{(n)}$ is projective (with bounded module maps as morphisms).

Corollary 3.9. *Let A be a unital C^*-algebra and let H be a Hilbert A-module. Suppose that there is an integer $n \geq 1$ and a surjective map $S \in B(A^{(n)}, H)$. Then H is projective (with bounded module maps as morphisms)*

Proof. Let H_1 and H_2 be two Hilbert A-modules and let $\varphi \in B(H_1, H_2)$ which is surjective. Suppose that $\psi \in B(H, H_2)$.

Since $A^{(n)}$ is self-dual, S^* must map H into $A^{(n)}$. In other words, $S \in L(A^{(n)}, H)$. By 3.4, there exists $T \in L(H, A^{(n)})$ such that

$$ST = \mathrm{id}_H. \tag{e 3.102}$$

Let $\varphi_1 \in B(A^{(n)}, H_2)$ be defined by

$$\varphi_1 = \varphi \circ S. \tag{e 3.103}$$

Then, by 3.7, $A^{(n)}$ is projective. There is $L \in B(A^{(n)}, H_1)$ such that

$$\varphi \circ L = \varphi_1. \tag{e 3.104}$$

Define $\varphi_2 = L \circ T$. Then $\varphi_2 \in B(H, H_1)$. Moreover,

$$\varphi \circ \varphi_2 \quad = \quad \varphi \circ L \circ T = \varphi \circ S \circ T = \varphi. \tag{e 3.105}$$

Hence H is projective (with bounded module maps as morphisms).

□

There are projective Hilbert modules (with bounded module maps as morphisms) for which $K(H)$ is not unital.

Theorem 3.10. *Let A be a separable C^*-algebra such that $LM(A \otimes \mathcal{K}) = M(A \otimes \mathcal{K})$. Then every countably generated Hilbert A-module is projective (with bounded module maps as morphisms)*

One needs the following lemma which the author could not locate a reference.

Lemma 3.11. *Let X be a Banach space and let H be a separable Banach space. Suppose that $T : X \to H$ is a surjective bounded linear map. Then there is a separable subspace $Y \subset X$ such that $TX = H$.*

Proof. Note that the Open Mapping Theorem applies here. From the open mapping theorem (or a proof of it), there is $\delta > 0$ for which $T(B(0, a))$ is dense in $O(0, a\delta)$ for any $a > 0$, where $B(0, a) = \{x \in X : \|x\| \leq a\}$ and $O(0, b) = \{h \in H : \|h\| < b\}$. For each rational number $r > 0$, since H is separable, one may find a countable set $E_r \subset B(0, r)$ such that $T(E_r)$ is dense in $O(0, r\delta)$. Let Y be the closed subspace generated by $\cup_{r \in \mathbb{Q}_+} E_r$.

Let $d = \delta/2$ and let $y_0 \in O(0,d)$. Then $T(Y \cap B(0,1/2))$ is dense in $O(0,d)$. Choose $\xi_1 \in Y \cap B(0,1/2)$ such that

$$\|y_0 - T\xi_1\| < \delta/2^2. \tag{e 3.106}$$

In particular,

$$y_1 = y_0 - T\xi_1 \in O(0, \delta/2^2). \tag{e 3.107}$$

Since $T(Y \cap B(0,1/2^2))$ is dense in $O(0,\delta/2^2)$, one obtains $\xi_2 \in Y \cap B(0,1/2^2)$ such that

$$\|y_1 - T\xi_2\| < \delta/2^3. \tag{e 3.108}$$

In other words,

$$y_2 = y_1 - T\xi_2 = y_0 - (T\xi_1 + T\xi_2) \in O(0, \delta/2^3). \tag{e 3.109}$$

Continuing this process, one obtains a sequence of elements $\{\xi_n\} \subset Y$ for which $\xi_n \in B(0,1/2^n)$ and

$$\|y_0 - (T\xi_1 + T\xi_2 + \cdots + T\xi_n)\| < \delta/2^{n+1}, \ \ n = 1,2,.... \tag{e 3.110}$$

Define $\xi_0 = \sum_{n=1}^{\infty} \xi_n$. Note that the sum converges in norm and therefore $\xi_0 \in Y$. By the continuity of T,

$$T\xi_0 = y_0. \tag{e 3.111}$$

This implies that $T(Y) \supset O(0,d)$. It follows that $T(Y) = H$. □

Proof of Theorem 3.10

Let H be a countably generated Hilbert A-module. Suppose that H_1 and H_2 are two Hilbert A-modules, suppose that $\varphi \in B(H_1, H_2)$ and $\psi \in B(H, H_2)$. Suppose also that φ is surjective.

Let $H_3 = \overline{\psi(H)}$. Then H_3 is countably generated. Since A is separable, H_3 is also a separable Banach space. By 3.11, there is a separable subspace $Y \subset H_1$ such that $TY = H_3$. Let H_4 be the Hilbert A-module generated by Y. Then H_4 is countably generated.

Let $H_0 = H_A \oplus H_4 \oplus H_3 \oplus H$. Then, by a result of Kasparov ([11]), $H_0 \cong H_A$. Define

$$\Psi(h_0 \oplus h_4 \oplus h_3 \oplus h) = \psi(h_4) \ \text{ and } \ \Phi(h_0 \oplus h_4 \oplus h_3 \oplus h) = \varphi(h) \tag{e 3.112}$$

for all $h_0 \in H_A, h_4 \in H_4, h_3 \in H_3$ and $h \in H$. Note that Ψ is from H_0 onto H_3.

By the assumption that $LM(A \otimes K) = M(A \otimes K)$ and by Theorem 1.5 of [14] and [11], $\Psi, \Phi \in L(H_0)$. It follows that $\varphi|_{H_4} \in L(H_4, H_3)$ and $\psi \in L(H, H_3)$. By 3.4, there exists $\varphi_1 \in L(H, H_4)$ such that

$$\varphi \circ \varphi_1 = \psi. \tag{e 3.113}$$

Lemma 3.12. *Let A be a C^*-algebra and let H be a Hilbert A-module. Let $H_0 \subset H$ be a Hilbert A-submodule. Suppose that $\{e_\alpha\}$ is an approximate identity for $K(H_0)$ and suppose that $\xi \in H$. Then*

$$\|\pi(\xi)\| = \lim_{\alpha} \|(1 - e_\alpha)(\xi)\|, \tag{e 3.114}$$

where $\pi : H \to H/H_0$ is the quotient map.

Proof. Note that

$$\|\pi(\xi)\| = \inf\{\|\xi+\zeta\| : \zeta \in H_0\}.$$

It follows from Lemma 2.13 of [15] that $K(H_0)$ may be regarded as a hereditary C^*-subalgebra of $K(H)$.

Let $\epsilon > 0$. There exists $\zeta \in H_0$ such that

$$\|\pi(\xi)\| \geq \|\xi+\zeta\| - \epsilon/2. \tag{e 3.115}$$

There exists α_0 such that

$$\|(1-e_\alpha)(\zeta)\| < \epsilon/4 \text{ for all } \alpha \geq \alpha_0. \tag{e 3.116}$$

Note that $0 \leq 1-e_\alpha \leq 1$ for all α. Therefore

$$\begin{aligned} \|\pi(\xi)\| &\geq \|\xi+\zeta\| - \epsilon/2 \geq \|(1-e_\alpha)(\xi+\zeta)\| - \epsilon/2 && \text{(e 3.117)}\\ &\geq \|(1-e_\alpha)(\xi)\| - \|(1-e_\alpha)(\zeta)\| - \epsilon/2 && \text{(e 3.118)}\\ &\geq \|(1-e_\alpha)(\xi)\| - \epsilon. && \text{(e 3.119)} \end{aligned}$$

Let $\epsilon \to 0$,

$$\|\pi(\xi)\| \geq \|(1-e_\alpha)(\xi)\| \text{ for all } \alpha \geq \alpha_0. \tag{e 3.120}$$

It follows that

$$\|\pi(\xi)\| \geq \lim_\alpha \|(1-e_\alpha)(\xi)\|. \tag{e 3.121}$$

Since $e_\alpha(\zeta) \in H_0$ for all α,

$$\|\pi(\xi)\| \leq \lim_\alpha \|(1-e_\alpha)(\xi)\|. \tag{e 3.122}$$

The lemma follows from the combination of (e 3.121) and (e 3.122).

□

Remark 3.13. Suppose that H_1 and H are Hilbert A-modules and $\varphi : H_1 \to H$ is a bounded surjective module map. Let $H_0 = \ker\varphi$. It is a Hilbert submodule of H_1. Let $\pi : H_1 \to H_1/H_0$ be the quotient map. It is a Banach space. There is a bounded linear map $\varphi' : H_1/H_0 \to H$ such that $\varphi' \circ \pi = \varphi$. Since φ' is one-to-one and onto, it has an inverse. In what follows denote by $\varphi^\sim : H \to H/H_0$ the inverse which is also bounded.

Let p be the open projection of $K(H_1)$ corresponding $K(H_0)$. Then H/H_0 may be identified with $(1-p)H$ which can also be made into a Banach A-module.

Lemma 3.14. *Let A be a C^*-algebra and let H be a Hilbert A-module. Suppose that $\xi_1, \xi_2, ..., \xi_n \in H$ and $e_1, e_2, ..., e_n \in A_+$ are in the center of A with $0 \leq e_i \leq 1$ $(i = 1, 2, ..., n)$ such that*

$$e_ie_j = e_je_i = 0 \text{ if } |i-j| \geq 2 \text{ and } \xi_ie_i = \xi_i, \ \ i = 1, 2, ..., n. \tag{e 3.123}$$

Then, for any $b \in A$,

$$\|\sum_{i=1}^n \xi_i b\| \leq 2 \max_{1\leq i\leq n} \|\xi_i\|\|b\|. \tag{e 3.124}$$

Proof. Let $F \in H^\sharp$ with $\|F\| \le 1$. Let p_i be the range projection of $F(\xi_i)^*F(\xi_i)$ in A^{**}, $i = 1, 2, ..., n$. Note that $p_ip_j = p_jp_i = 0$ if $|i-j| \ge 2$.

Define

$$C_0 = \begin{pmatrix} F(\xi_2) & F(\xi_4) & \cdots & F(\xi_{2k}) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} F(\xi_1) & F(\xi_3) & \cdots & F(\xi_{2k-1}) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} p_2b_2 & 0 & \cdots & 0 \\ p_4b_4 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ p_{2k}b_{2k} & 0 & \cdots & 0 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} p_1b_1 & 0 & \cdots & 0 \\ p_3b_3 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ p_{2k-1}b_{2k-1} & 0 & \cdots & 0 \end{pmatrix}.$$

Here if n is even, then $2k = n$, if n is odd, then $n = 2k-1$ and $\xi_{2k} = 0$. One estimates that

$$\begin{aligned}
\|F(\sum_{i=1}^n \xi_i b)\| &= \|\sum_{i=1}^n F(\xi_i)p_ib\| && \text{(e 3.125)}\\
&\le \|\sum_{i=odd} F(\xi_i)p_ib\| + \|\sum_{i=even} F(\xi_i)p_ib\| && \text{(e 3.126)}\\
&= \|C_1B_1\| + \|C_0B_0\| \le (\|C_1^*C_1\|\|B_1^*B_1\|)^{1/2} + (\|C_0^*C_0\|\|B_0^*B_0\|)^{1/2} && \text{(e 3.127)}\\
&= (\|C_1C_1^*\|\|B_1^*B_1\|)^{1/2} + (\|C_0C_0^*\|\|B_0^*B_0\|)^{1/2} && \text{(e 3.128)}\\
&\le (\|\sum_{i=odd} F(\xi_i)F(\xi_i)^*\|\|\sum_{i=odd} b^*p_ib\|)^{1/2} && \text{(e 3.129)}\\
&\quad +(\|\sum_{i=even} F(\xi_i)F(\xi_i)^*\|\|\sum_{i=even}^n b^*p_ib\|)^{1/2} && \text{(e 3.130)}\\
&= (\|\sum_{i=odd} e_iF(\xi_i)F(\xi_i)^*e_i\|\|b^*(\sum_{i=odd}^n p_i)b\|)^{1/2} && \text{(e 3.131)}\\
&\quad +(\|\sum_{i=even} e_iF(\xi_i)F(\xi_i)^*e_i\|\|b^*(\sum_{i=even}^n p_i)b\|)^{1/2} && \text{(e 3.132)}\\
&\le ((\max_{i=odd}\|F(\xi_i)F(\xi_i)^*\|)\|\sum_{i=odd} e_i\|\|b^*b\|)^{1/2} && \text{(e 3.133)}\\
&\quad +((\max_{i=even}\|F(\xi_i)F(\xi_i)^*\|)\|\sum_{i=even} e_i\|\|b^*b\|)^{1/2} && \text{(e 3.134)}\\
&\le 2\max_{1\le i\le n}\|\xi_i\|(\|b^*b\|)^{1/2} && \text{(e 3.135)}\\
& && \text{(e 3.136)}
\end{aligned}$$

It follows that

$$\|\sum_{i=1}^n \xi_ib_i\| \le 2\max_{1\le i\le n}\|\xi_i\|\|b\| \qquad \text{(e 3.137)}$$

□

Remark 3.15. In the lemma above, if $e_1, e_2, ..., e_n$ are mutually orthogonal, then the number 2 in (e 3.124) can be replaced by 1.

Definition 3.16. Let A be a C^*-algebra. An approximate identity $\{e_n\}$ is said to be a *sequential central* approximate identity, if $\{e_n\}$ is a sequence and each e_n is in the center of A.

Theorem 3.17. *Let A be a unital C^*-algebra, let $a \in A \setminus \{0\}$ and let $H = \overline{aA}$. Suppose that $K(H)$ has a sequential central approximate identity. Then H is a projective Hilbert A-module (with bounded module maps as morphisms).*

Moreover, if H_1 and H_2 are two Hilbert A-modules, $\varphi \in B(H_1, H_2)$ is surjective and if $\psi \in B(H, H_2)$. Then, for any $\epsilon > 0$, there exists $T \in B(H, H_1)$ with

$$\|T\| \le 2\|\varphi^\sim \circ \psi\| + \epsilon$$

such that

$$\varphi \circ T = \psi.$$

In the case that $K(H)$ admits a central approximate identity consisting of a sequence of projections, one can choose $T \in B(H, H_2)$ such that

$$\|T\| \le \|\varphi^\sim \circ \psi\| + \epsilon.$$

Proof. Since $\overline{aAa}$ has a sequential central approximate identity, $\overline{aAa}$ contains a strictly positive element x which is in the center. One may assume that $a = x$ and $\mathrm{sp}(a) = [0, 1]$. Let $f_n \in C_0((0, 1])$ be such that $0 \le f_n \le 1$, $f_n(t) = 1$ if $t \in [1/2^n, 1]$, $f_n(t) = 0$ if $t \in [0, 3/2^{n+2}]$ and $f(t)$ is linear in $[3/2^{n+2}, 1/2^n]$, $n = 1, 2, ...$, and let $g_n \in C_0((0, 1])$ be such that $0 \le g_n \le 1$, $g_n(t) = 1$ if $t \in [1/2^{n+2} - 1/2n2^{n+2}, 1/2^n + 1/2n2^{n+2}]$, $g_n(t) = 0$ if $t \notin [1/2^{n+2} - 1/n2^{n+2}, 1/2^n + 1/n2^{n+2}]$ and $g_n(t)$ is linear in $[1/2^{n+2} - 1/n2^{n+2}, 1/2^{n+2} - 1/2n2^{n+2}]$ and $[1/2^n + 1/2n2^{n+2}, 1/2^n + 1/n2^{n+2}]$, $n = 1, 2, ...$

Define $e_n = f_{n+1}(a) - f_n(a)$ and $d_1 = f_1(a)$, $d_n = g_n(a)$, $n = 2, 3, ...$ One has that

$$e_n d_n = d_n e_n = e_n, d_n d_m = d_m d_n = 0 \text{ if } |n - m| \ge 2, n, m = 1, 2, ... \tag{e 3.138}$$

Suppose that H_1 and H_2 are two Hilbert A-modules and $\varphi \in B(H_1, H_2)$ is surjective. Suppose also that there is $\psi \in B(H, H_2)$. Denote by H_3 the closure of $\psi(H)$. Then H_3 is countably generated.

Let p be the open projection of $K(H_1)$ associated with the Hilbert submodule $\ker\varphi$. Let $\varphi' : H_1/\ker\varphi \to H_2$ be the one-to-one and onto bounded module map such that

$$\varphi'(\pi(x)) = \varphi(x) \text{ for all } x \in H. \tag{e 3.139}$$

Denote by $\varphi^\sim$ the inverse of φ' which is also a bounded module map. There is $x_i \in H_1$ such that

$$\varphi(x_i) = \psi(e_i), \ \ i = 1, 2, ... \tag{e 3.140}$$

Let $\{p_\alpha\}$ be an approximate identity for $K(\ker\varphi)$. By 2.12 of [15], one may view $K(\ker\varphi) \subset K(H_1)$. Then, by 3.12

$$\|\pi(x_i)\| = \inf_\alpha \|(1 - p_\alpha)x_i\|, \ \ i = 1, 2, ... \tag{e 3.141}$$

For any $\epsilon > 0$. Choose p_n so that

$$\|(1 - p_n)x_n\| \le \|\pi(x_n)\| + \epsilon/2^{n+1} = \|\varphi^\sim(e_i)\| + \epsilon/2^{n+1}, \ \ n = 1, 2, ... \tag{e 3.142}$$

Put $\xi_n = (1-p_n)x_n d_n$. Note that $\varphi(\xi_n) = e_n d_n = e_n$, $n = 1, 2, ...$ For each n, and $b \in A$, define

$$T(f_n(a)b) = \sum_{i=1}^{n} \xi_i b \text{ for all } b \in A. \tag{e 3.143}$$

By applying 3.14

$$\|T(\sum_{i=k}^{n+k} e_i b)\| \leq 2 \max_{k \leq i \leq n+k} \|\xi_i\| \|(f_{n+k}(a) - f_n(a))b\| \tag{e 3.144}$$

$$\leq 2(\|\varphi^{\sim}\| + \sum_{i=1}^{n} \epsilon/2^{i+1})\|(f_{n+k}(a) - f_k(a))b\|. \tag{e 3.145}$$

Therefore, since $\{f_m(a) : m = 1, 2, ...\}$ forms an approximate identity for $\overline{aAa}$, for any $b \in \overline{aA}$,

$$\lim_{k\to\infty} \|\sum_{i=k}^{k+n} \xi_i b\| \leq 2(\|\varphi^{\sim}\| + 1) \lim_{k\to\infty} \|(f_{n+k}(a) - f_k(a))b\| = 0. \tag{e 3.146}$$

Thus, one defines, for each $b \in B$,

$$T(b) = \sum_{n=1}^{\infty} \xi_n b. \tag{e 3.147}$$

By (e 3.145),

$$\|T(b)\| \leq 2(\|\varphi^{\sim}\| + \epsilon)\|b\| \text{ for all } b \in \overline{aA}. \tag{e 3.148}$$

So T is well-define map in $B(H, H_2)$. One verifies that

$$\varphi \circ T(b) = \varphi \circ T(\sum_{n=1}^{\infty} e_n b) = \varphi(\sum_{n=1}^{\infty} \xi_n b) \tag{e 3.149}$$

$$= \varphi(\sum_{i=1}^{\infty} (1-p_n)x_n d_n b) = \sum_{n=1}^{\infty} \varphi(x_n)b \tag{e 3.150}$$

$$= \sum_{n=1}^{\infty} \psi(e_n)b = \sum_{n=1}^{\infty} \psi(e_n b) \tag{e 3.151}$$

$$= \psi(b). \tag{e 3.152}$$

□

Corollary 3.18. *Let A be a C^*-algebra and let $x_1, x_2, .., x_n \in A$. Suppose that $H_i = \overline{x_i A}$ and $K(H_i)$ admits a sequential central approximate identity, $i = 1, 2, ..., n$. Then*

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

is a projective Hilbert A-module (with bounded module maps as morphisms).

References

[1] C. Akemann and F. Shultz, *Perfect C^*-algebras*, Mem. Amer. Math. Soc. 55 (1985), no. 326, xiii+117 pp

[2] B. Blackadar, *Comparison theory for simple C^*-algebras. Operator algebras and applications*, Vol. 1, 21–54, London Math. Soc. Lecture Note Ser., 135, Cambridge Univ. Press, Cambridge, 1988.

[3] L. G. Brown, *Close hereditary C^*-subalgebra and the structure of quasi-multipliers*, Preprint (MSRI 11211-85).

[4] L. G. Brown, *Homotopy of projection in C^*-algebras of stable rank one*, Recent advances in operator algebras (Orlans, 1992). Astrisque No. 232 (1995), 115–120.

[5] L. G. Brown and G. K. Pedersen, *On the geometry of the unit ball of a C^*-algebra*, J. Reine Angew. Math. **469** (1995), 113–147.

[6] N. Brown and A. Ciuperca, *Isomorphism of Hilbert modules over stably finite C^*-algebras*, J. Funct. Anal. **257** (2009), 332–339.

[7] N. Brown, F. Perera and A. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras*, J. Reine Angew. Math. **621** (2008), 191–211.

[8] K. Coward, G. Elliott and C. Ivanescu, *The Cuntz semigroup as an invariant for C^*-algebras*, J. Reine Angew. Math. **623** (2008), 161–193.

[9] J. Cuntz, *The structure of multiplication and addition in simple C^*-algebras*, Math. Scand. **40** (1977), 215–233.

[10] J. Cuntz, *Dimension functions on simple C^*-algebras*, Math. Ann. **233** (1978), 145–153.

[11] G. Kasparov, *Hilbert C^*-modules: theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133–150.

[12] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), 637–666.

[13] H. Lin, *Equivalent open projections and corresponding hereditary C^*-subalgebras*, J. London Math. Soc. **41** (1990), 295–301.

[14] H. Lin, *Bounded module maps and pure completely positive maps*, J. Operator Theory **26** (1991), 121–138.

[15] H. Lin, *Injective Hilbert C^*-modules*, Pacific J. Math. **154** (1992), 131–164.

[16] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xii+320 pp. ISBN: 981-02-4680-3.

[17] G. K. Pedersen, *Unitary extensions and polar decompositions in a C^*-algebra*, J. Operator Theory **17** (1987), 357–364.

[18] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), 255–269.

[19] M. Rieffel, *The homotopy groups of the unitary groups of noncommutative tori*, J. Operator Theory **17** (1987), 237–254.

[20] A. Toms, *On the classification problem for nuclear C^*-algebras.* Ann. of Math. **167** (2008), 1029–1044.

[21] S. Zhang, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras. I*, Pacific J. Math. **155** (1992), 169–197

email: hlin@uoregon.edu